

# GENERATORS OF VON NEUMANN ALGEBRAS ASSOCIATED WITH SPECTRAL MEASURES

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**ABSTRACT.** Let  $\mathcal{P}_E$  be the set of all values of a spectral measure  $E$  and  $\mathcal{A}(\mathcal{P}_E)$  be the smallest von Neumann algebra containing  $\mathcal{P}_E$ . We give a simple description of all sets of generators of  $\mathcal{A}(\mathcal{P}_E)$  in terms of the integrals with respect to  $E$ . The treatment covers not only the case of generators belonging to  $\mathcal{A}(\mathcal{P}_E)$ , but also the case of (possibly unbounded) generators affiliated with this algebra.

Let  $S$  be a measurable space with a  $\sigma$ -algebra  $\Sigma$  of measurable sets,  $\mathfrak{H}$  be a Hilbert space, and  $L(\mathfrak{H})$  be the algebra of all bounded everywhere defined linear operators in  $\mathfrak{H}$ . Recall [2] that a map  $E$  from  $\Sigma$  to the set of orthogonal projections on  $\mathfrak{H}$  is called a spectral measure for  $(S, \mathfrak{H})$  if it is countably additive with respect to the strong operator topology on  $L(\mathfrak{H})$  and  $E(S)$  is the identity operator in  $\mathfrak{H}$ . Let  $\mathcal{P}_E$  denote the set of all projections  $E(A)$  with  $A \in \Sigma$ , and let  $\mathcal{A}(\mathcal{P}_E)$  be the von Neumann algebra generated by  $\mathcal{P}_E$ , i.e., the smallest von Neumann algebra containing  $\mathcal{P}_E$ . In this paper, we give a simple description of all sets of generators of  $\mathcal{A}(\mathcal{P}_E)$  in terms of the integrals with respect to  $E$ .

This problem arises naturally when considering self-adjoint extensions of Schrödinger operators with singular potentials commuting with a given set of symmetries [5]. In such quantum-mechanical applications, it is convenient to consider not only generators belonging to the considered algebra, but also more general operators which are affiliated with it and may be unbounded. In this paper, we shall work in such a more general setting.

We say that an operator  $T$  in  $\mathfrak{H}$  with the domain  $D_T$  commutes with  $R \in L(\mathfrak{H})$  if  $R\Psi \in D_T$  and  $RT\Psi = TR\Psi$  for any  $\Psi \in D_T$ . Given a set  $\mathcal{X}$  of closed densely defined operators in  $\mathfrak{H}$ , let  $\mathcal{X}'$  denote its commutant, i.e., the subalgebra of  $L(\mathfrak{H})$  consisting of all operators commuting with every element of  $\mathcal{X}$ . We denote by  $\mathcal{X}^*$  the set consisting of the adjoints of the elements of  $\mathcal{X}$ . The set  $\mathcal{X}$  is called involutive if  $\mathcal{X}^* = \mathcal{X}$ . If a densely defined operator  $T$  commutes with  $R \in L(\mathfrak{H})$ , then its adjoint  $T^*$  commutes with  $R^*$ . Indeed, if  $\Psi \in D_{T^*}$  and  $\Phi = T^*\Psi$ , then we have  $\langle T\Psi', \Psi \rangle = \langle \Psi', \Phi \rangle$  for any  $\Psi' \in D_T$  and, hence

$$\langle T\Psi', R^*\Psi \rangle = \langle TR\Psi', \Psi \rangle = \langle R\Psi', \Phi \rangle = \langle \Psi', R^*\Phi \rangle, \quad \Psi' \in D_T,$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product on  $\mathfrak{H}$ . This means that  $R^*\Psi \in D_{T^*}$  and  $T^*R^*\Psi = R^*T^*\Psi$ , i.e.,  $T^*$  commutes with  $R^*$ . It follows that

$$(1) \quad (\mathcal{X}')^* = (\mathcal{X}^*)'.$$

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Recall [4] that a subalgebra  $\mathcal{M}$  of  $L(\mathfrak{H})$  is called a von Neumann algebra if it is involutive and coincides with its bicommutant  $\mathcal{M}''$ . By the well-known von Neumann's bicommutant theorem (see, e. g., [4], Sec. I.3.4, Corollaire 2), an involutive subalgebra  $\mathcal{M}$  of  $L(\mathfrak{H})$  is a von Neumann algebra if and only if it contains the identity operator and is closed in the strong operator topology. It follows from (1) that  $\mathcal{X}'$  is an involutive subalgebra of  $L(\mathfrak{H})$  for any involutive set  $\mathcal{X}$  of closed densely defined operators in  $\mathfrak{H}$ . Moreover, as shown by the next lemma,  $\mathcal{X}'$  is always strongly closed and, therefore, is a von Neumann algebra for involutive  $\mathcal{X}$  by the bicommutant theorem.

**Lemma 1.** *Let  $\mathcal{X}$  be a set of densely defined closed operators in  $\mathfrak{H}$ . Then  $\mathcal{X}'$  is a strongly closed subset of  $L(\mathfrak{H})$ .*

*Proof.* Given an operator  $T$  in  $\mathfrak{H}$ , let  $C_T$  denote the set of all elements of  $L(\mathfrak{H})$  commuting with  $T$ . Since  $\mathcal{X}' = \bigcap_{T \in \mathcal{X}} C_T$ , it suffices to prove that  $C_T$  is strongly closed for any closed  $T$ . Let  $R$  belong to the strong closure of  $C_T$ . For every  $\Psi_1, \Psi_2 \in \mathfrak{H}$  and  $n = 1, 2, \dots$ , the set

$$W_{\Psi_1, \Psi_2, n} = \{\tilde{R} \in L(\mathfrak{H}) : \|(\tilde{R} - R)\Psi_i\| < 1/n, i = 1, 2\}$$

is a strong neighborhood of  $R$  and, hence, has a nonempty intersection with  $C_T$ . Fix  $\Psi \in D_T$  and choose  $R_n \in C_T \cap W_{\Psi, T\Psi, n}$  for each  $n$ . Then  $R_n\Psi \rightarrow R\Psi$  and  $R_n T\Psi \rightarrow RT\Psi$  in  $\mathfrak{H}$ . As  $R_n$  commute with  $T$ , we have  $R_n\Psi \in D_T$  and  $R_n T\Psi = TR_n\Psi$  for all  $n$ . In view of the closedness of  $T$ , it follows that  $R\Psi \in D_T$  and  $TR\Psi = RT\Psi$ , i.e.,  $R \in C_T$ . The lemma is proved.  $\square$

A closed densely defined operator  $T$  in  $\mathfrak{H}$  is called affiliated [6] with a von Neumann algebra  $\mathcal{M}$  if  $T$  commutes with every element of  $\mathcal{M}'$ . If  $\mathcal{X}$  is a set of closed densely defined operators in  $\mathfrak{H}$ , then every element of  $\mathcal{X}$  is obviously affiliated with the algebra  $\mathcal{A}(\mathcal{X}) = (\mathcal{X} \cup \mathcal{X}^*)''$ . In fact, the latter is the smallest von Neumann algebra with this property. The algebra  $\mathcal{A}(\mathcal{X})$  will be called the von Neumann algebra generated by  $\mathcal{X}$ , and  $\mathcal{X}$  will be referred to as a set of generators of  $\mathcal{A}(\mathcal{X})$ . If  $\mathcal{X} \subset L(\mathfrak{H})$ , then  $\mathcal{A}(\mathcal{X})$  is just the smallest von Neumann algebra containing  $\mathcal{X}$ . Clearly, a closed densely defined operator  $T$  is affiliated with a von Neumann algebra  $\mathcal{M}$  if and only if  $\mathcal{A}(T) \subset \mathcal{M}$  (here and subsequently, we write  $\mathcal{A}(T)$  instead of  $\mathcal{A}(\{T\})$ , where  $\{T\}$  is the one-point set containing  $T$ ).

Let  $E$  be a spectral measure for  $(\mathfrak{H}, S)$ . Then  $E(A_1 \cap A_2) = E(A_1)E(A_2)$  for any measurable  $A_1, A_2 \subset S$  (see [2], Sec. 5.1, Theorem 1) and, hence,  $E(A_1)$  and  $E(A_2)$  commute. This implies that the algebra  $\mathcal{A}(\mathcal{P}_E)$  is Abelian. For any  $\Psi \in \mathfrak{H}$ , the finite positive measure  $E_\Psi$  on  $S$  is defined by setting  $E_\Psi(A) = \langle E(A)\Psi, \Psi \rangle$  for any measurable  $A$ . Given an  $E$ -measurable<sup>1</sup> function complex function  $f$  on  $S$ , the integral  $J_f^E$  of  $f$  with respect to  $E$  is defined as the unique linear operator in  $\mathfrak{H}$  such that

$$(2) \quad D_{J_f^E} = \left\{ \Psi \in \mathfrak{H} : \int |f(s)|^2 dE_\Psi(s) < \infty \right\}$$

$$\langle \Psi, J_f^E \Psi \rangle = \int f(s) dE_\Psi(s), \quad \Psi \in D_{J_f^E}.$$

<sup>1</sup>As usual, a complex function  $f$  defined on  $S$  is called measurable if the set  $f^{-1}(A)$  is measurable for any Borel subset  $A$  of  $\mathbb{C}$ . A complex function is called  $E$ -measurable on  $S$  if it is defined  $E$ -almost everywhere ( $E$ -a.e.) and coincides  $E$ -a.e. with a measurable function.

For every normal<sup>2</sup> operator  $T$ , there is a unique spectral measure  $\mathcal{E}_T$  on  $\mathbb{C}$  such that  $J_{\text{id}_{\mathbb{C}}}^{\mathcal{E}_T} = T$ , where  $\text{id}_{\mathbb{C}}$  is the identity function on  $\mathbb{C}$ . The operators  $\mathcal{E}_T(A)$ , where  $A$  is a Borel subset of  $\mathbb{C}$ , are called the spectral projections of  $T$ . If  $f$  is an  $\mathcal{E}_T$ -measurable complex function on  $\mathbb{C}$ , then the operator  $J_f^{\mathcal{E}_T}$  is also denoted as  $f(T)$ .

Recall that a topological space  $S$  is called a Polish space if its topology can be induced by a metric that makes  $S$  a separable complete space. A measurable space  $S$  is called a standard Borel space if its measurable structure can be induced by a Polish topology on  $S$ . A spectral measure  $E$  on a measurable space  $S$  is called standard if there is a measurable set  $S' \subset S$  such that  $E(S \setminus S') = 0$  and  $S'$ , considered as a measurable subspace of  $S$ , is a standard Borel space.

Let  $\{f_\iota\}_{\iota \in I}$  be a family of maps and  $S$  be a set contained in the domains of  $f_\iota$  for all  $\iota \in I$ . The family  $\{f_\iota\}_{\iota \in I}$  is said to separate points of  $S$  if for any two distinct elements  $s_1$  and  $s_2$  of  $S$ , there is  $\iota \in I$  such that  $f_\iota(s_1) \neq f_\iota(s_2)$ .

**Definition 2.** Let  $S$  be a measurable space and  $E$  be a spectral measure on  $S$ . A family  $\{f_\iota\}_{\iota \in I}$  of maps defined  $E$ -a.e. on  $S$  is said to be  $E$ -separating on  $S$  if  $I$  is countable and  $\{f_\iota\}_{\iota \in I}$  separates points of  $S \setminus N$  for some  $E$ -null set  $N$ .

Our aim is to prove the next result.

**Theorem 3.** Let  $S$  be a measurable space,  $\mathfrak{H}$  be a separable Hilbert space, and  $E$  be a standard spectral measure for  $(S, \mathfrak{H})$ . The algebra  $\mathcal{A}(\mathcal{P}_E)$  is generated by a set  $\mathcal{X}$  of closed densely defined operators in  $\mathfrak{H}$  if and only if the following conditions hold

- (1) For every  $T \in \mathcal{X}$ , there is an  $E$ -measurable complex function  $f$  on  $S$  such that  $T = J_f^E$ .
- (2) There is an  $E$ -separating family  $\{f_\iota\}_{\iota \in I}$  of  $E$ -measurable complex functions on  $S$  such that  $J_{f_\iota}^E \in \mathcal{X}$  for all  $\iota \in I$ .

*Remark 4.* Every Abelian von Neumann algebra  $\mathcal{M}$  acting in a separable Hilbert space  $\mathfrak{H}$  is equal to  $\mathcal{A}(\mathcal{P}_E)$  for a suitable standard spectral measure  $E$ . Indeed, by Théorème 2 of Sec. II.6.2 in [4], there are a finite measure  $\nu$  on a compact metrizable space  $S$ , a  $\nu$ -measurable family  $\mathfrak{S}(s)$  of Hilbert spaces, and a unitary operator  $V: \mathfrak{H} \rightarrow \int_S^\oplus \mathfrak{S}(s) d\nu(s)$  such that  $\mathcal{M}$  coincides with the set of all operators  $V^{-1}\mathcal{T}_gV$ , where  $\mathcal{T}_g$  is the operator of multiplication by a  $\nu$ -measurable  $\nu$ -essentially bounded function  $g$  on  $S$ . Then we can define  $E$  by setting  $E(A) = V^{-1}\mathcal{T}_{\chi_A}V$  for any Borel set  $A \subset S$ , where  $\chi_A$  is the characteristic function of  $A$ .

**Example 5.** Let  $\nu$  be a positive measure on a measurable space  $S$  and let the spectral measure  $E$  in the Hilbert space  $L^2(S, \nu)$  be given by  $E(A) = \mathcal{T}_{\chi_A}$ , where  $\mathcal{T}_{\chi_A}$  is the operator of multiplication by the characteristic function of  $\chi_A$  of  $A$ . Then  $\mathcal{A}(\mathcal{P}_E)$  is identified with the algebra  $L^\infty(S, \nu)$  (see Lemma 9 below) acting by multiplication on  $L^2(S, \nu)$ . If  $\nu$  is standard, then it follows from Theorem 3 that a set  $\mathcal{C} \subset L^\infty(S, \nu)$  generates  $L^\infty(S, \nu)$  if and only if  $\mathcal{C}$  contains a  $\nu$ -separating family on  $S$ . This statement can be viewed as an analogue of the Stone-Weierstrass theorem in the setting of  $L^\infty$ -spaces.

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<sup>2</sup>Recall that a closed densely defined linear operator  $T$  in a Hilbert space  $\mathfrak{H}$  is called normal if the operators  $TT^*$  and  $T^*T$  have the same domain of definition and coincide thereon.

We say that two sets  $\mathcal{X}$  and  $\mathcal{Y}$  of closed densely defined operators in  $\mathfrak{H}$  are equivalent if  $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{Y})$ . We say that  $\mathcal{X}$  is equivalent to a closed densely defined operator  $T$  if  $\mathcal{X}$  is equivalent to the one-point set  $\{T\}$ .

**Example 6.** Let  $\Lambda \subset \mathbb{C}$  be a set having an accumulation point in  $\mathbb{C}$  and let  $f_\lambda(z) = e^{\lambda z}$  for  $\lambda \in \Lambda$  and  $z \in \mathbb{C}$ . Clearly, we can choose a countable set  $\Lambda_0 \subset \Lambda$  that has an accumulation point in  $\mathbb{C}$ . If  $f_\lambda(z) = f_\lambda(z')$  for some  $z, z' \in \mathbb{C}$  and all  $\lambda \in \Lambda_0$ , then we have  $z = z'$  by the uniqueness theorem for analytic functions and, therefore, the family  $\{f_\lambda\}_{\lambda \in \Lambda_0}$  separates points of  $\mathbb{C}$ . Theorem 3 therefore implies that  $\mathcal{P}_E$  is equivalent to  $\{J_{f_\lambda}^E\}_{\lambda \in \Lambda}$  for any spectral measure  $E$  on  $\mathbb{C}$ . If  $T$  is a normal operator, then applying Theorem 3 to the identity function on  $\mathbb{C}$  yields the well-known fact (see, e.g., Theorem 3 of Sec. 6.6 in [2]) that  $T$  is equivalent to  $\mathcal{P}_{\mathcal{E}_T}$ . This implies that  $T$  is equivalent to the set of all operators  $e^{\lambda T}$  with  $\lambda \in \Lambda$ .

The rest of the paper is devoted to the proof of Theorem 3.

**Lemma 7.** *Let  $\{\mathcal{X}_\iota\}_{\iota \in I}$  and  $\{\mathcal{Y}_\iota\}_{\iota \in I}$  be families of sets of closed densely defined operators in  $\mathfrak{H}$  and let  $\mathcal{X} = \bigcup_{\iota \in I} \mathcal{X}_\iota$  and  $\mathcal{Y} = \bigcup_{\iota \in I} \mathcal{Y}_\iota$ . If  $\mathcal{X}_\iota$  and  $\mathcal{Y}_\iota$  are equivalent for every  $\iota \in I$ , then  $\mathcal{X}$  and  $\mathcal{Y}$  are equivalent.*

*Proof.* Set  $\mathcal{M}_\iota = (\mathcal{X}_\iota \cup \mathcal{X}_\iota^*)'$  and  $\mathcal{M} = (\mathcal{X} \cup \mathcal{X}^*)'$ . Then we have  $\mathcal{M} = \bigcap_{\iota \in I} \mathcal{M}_\iota$ . It follows that  $\mathcal{M}' = \mathcal{A}(\mathcal{X})$  coincides with the von Neumann algebra generated by  $\bigcup_{\iota \in I} \mathcal{M}_\iota' = \bigcup_{\iota \in I} \mathcal{A}(\mathcal{X}_\iota)$  (see [4], Sec. I.1.1, Proposition 1). Analogously,  $\mathcal{A}(\mathcal{Y})$  is the von Neumann algebra generated by  $\bigcup_{\iota \in I} \mathcal{A}(\mathcal{Y}_\iota)$ . Since  $\mathcal{A}(\mathcal{X}_\iota) = \mathcal{A}(\mathcal{Y}_\iota)$  for all  $\iota$ , it follows that  $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{Y})$ . The lemma is proved.  $\square$

**Lemma 8.** *Let  $\mathfrak{H}$  be a separable Hilbert space and  $\mathcal{X}$  be a set of closed densely defined operators in  $\mathfrak{H}$ . Then there is a countable subset  $\mathcal{X}_0$  of  $\mathcal{X}$  which is equivalent to  $\mathcal{X}$ .*

*Proof.* We first note that every subset of  $L(\mathfrak{H})$  is separable in the strong topology. Indeed, for any  $M \subset L(\mathfrak{H})$ , we have  $M = \bigcup_{n=1}^{\infty} M \cap \mathcal{B}_n$ , where  $\mathcal{B}_n = \{T \in L(\mathfrak{H}) : \|T\| \leq n\}$  is the ball of radius  $n$  in  $L(\mathfrak{H})$ . Since  $\mathfrak{H}$  is separable,  $\mathcal{B}_n$  endowed with the strong topology is a separable metrizable space for any  $n$  (see, e.g., [4], Sec. I.3.1). This implies that  $M \cap \mathcal{B}_n$  is separable for any  $n$  and, hence,  $M$  is separable in the strong topology.

Let  $\mathfrak{A} = \bigcup_{\mathcal{Y} \subset \mathcal{X}} \mathcal{A}(\mathcal{Y})$ , where  $\mathcal{Y}$  runs through all finite subsets of  $\mathcal{X}$ . Obviously,  $\mathcal{A}(T)$  is equivalent to  $T$  for any closed densely defined operator  $T$  and, therefore, Lemma 7 implies that  $\mathcal{X}$  is equivalent to  $\bigcup_{T \in \mathcal{X}} \mathcal{A}(T)$ . Since the latter set is contained in  $\mathfrak{A}$  and  $\mathfrak{A} \subset \mathcal{A}(\mathcal{X})$ , we conclude that  $\mathfrak{A}$  is equivalent to  $\mathcal{X}$ . We now note that  $\mathfrak{A}$  is an involutive subalgebra of  $L(\mathfrak{H})$  containing the identity operator and, therefore, is strongly dense in  $\mathfrak{A}'' = \mathcal{A}(\mathcal{X})$  ([4], Sec. I.3.4, Lemma 6). Let  $\mathfrak{R}$  be a strongly dense countable subset of  $\mathfrak{A}$ . For any  $R \in \mathfrak{R}$ , we choose a finite set  $\mathcal{Y}_R \subset \mathcal{X}$  such that  $R \in \mathcal{A}(\mathcal{Y}_R)$  and put  $\mathcal{X}_0 = \bigcup_{R \in \mathfrak{R}} \mathcal{Y}_R$ . Clearly,  $\mathcal{X}_0$  is a countable set. The algebra  $\mathcal{A}(\mathcal{X}_0)$  is strongly dense in  $\mathcal{A}(\mathcal{X})$  because it contains  $\mathfrak{R}$ . On the other hand,  $\mathcal{A}(\mathcal{X}_0)$  is a von Neumann algebra and, therefore, is strongly closed. We hence have  $\mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{X})$ , i.e.,  $\mathcal{X}_0$  is equivalent to  $\mathcal{X}$ . The lemma is proved.  $\square$

Let  $E$  be a spectral measure for  $(S, \mathfrak{H})$ . For any  $E$ -measurable complex function  $f$  on  $S$ , the operator  $J_f^E$  is normal, and we have  $J_f^E = J_{\bar{f}}^{E*}$ , where  $\bar{f}$  is the complex conjugate function of  $f$ . For any  $E$ -measurable  $f$  and  $g$  on  $S$ , we have

$$(3) \quad J_{fg}^E = \overline{J_f^E J_g^E}, \quad J_{f+g}^E = \overline{J_f^E + J_g^E},$$

where the bar means closure (see Theorem 7 of Sec. 5.4 in [2]). The operator  $J_f^E$  is everywhere defined and bounded if and only if  $f$  is  $E$ -essentially bounded. In this case, we have

$$(4) \quad \|J_f^E\| = E\text{-ess sup}_{s \in S} |f(s)|.$$

Let  $S$  and  $S'$  be measurable spaces,  $\mathfrak{H}$  be a Hilbert space,  $E$  be a spectral measure for  $(S, \mathfrak{H})$ , and  $\varphi: S \rightarrow S'$  be an  $E$ -measurable map. We denote by  $\varphi_*E$  the push-forward of  $E$  under  $\varphi$ . By definition (see [2], Sec. 5.4), this means that  $\varphi_*E$  is the spectral measure for  $(S', \mathfrak{H})$  such that

$$\varphi_*E(A) = E(\varphi^{-1}(A))$$

for any measurable  $A \subset S'$ . If  $f$  is a  $\varphi_*E$ -measurable complex function on  $S'$ , then  $f \circ \varphi$  is an  $E$ -measurable function on  $S$ , and we have

$$(5) \quad J_f^{\varphi_*E} = J_{f \circ \varphi}^E.$$

Let  $\varphi$  be an  $E$ -measurable complex function on  $S$ . Formula (5) with  $f = \text{id}_{\mathbb{C}}$  yields  $J_{\text{id}_{\mathbb{C}}}^{\varphi_*E} = J_{\varphi}^E$ . In view of the uniqueness of  $\mathcal{E}_{J_{\varphi}^E}$ , this means that

$$(6) \quad \mathcal{E}_{J_{\varphi}^E} = \varphi_*E.$$

By (5), it follows that  $f \circ \varphi$  is  $E$ -measurable and

$$(7) \quad f(J_{\varphi}^E) = J_f^{\varphi_*E} = J_{f \circ \varphi}^E$$

for any  $\mathcal{E}_{J_{\varphi}^E}$ -measurable complex function  $f$  on  $\mathbb{C}$ .

It follows from Theorem 5.6.18 in [6] that a closed densely defined operator  $T$  in  $\mathfrak{H}$  is normal if and only if the algebra  $\mathcal{A}(T)$  is Abelian.

The next lemma implies, in particular, that condition (1) of Theorem 3 holds if and only if every element of  $\mathcal{X}$  is affiliated with  $\mathcal{A}(\mathcal{P}_E)$ .

**Lemma 9.** *Let  $S$  be a measurable space,  $\mathfrak{H}$  be a separable Hilbert space, and  $E$  be a spectral measure for  $(\mathfrak{H}, S)$ . Then  $\mathcal{A}(\mathcal{P}_E)$  coincides with the set of all  $J_f^E$ , where  $f$  is an  $E$ -measurable  $E$ -essentially bounded complex function on  $S$ . A closed densely defined operator  $T$  in  $\mathfrak{H}$  is equal to  $J_f^E$  for an  $E$ -measurable complex function  $f$  on  $S$  if and only if*

$$(8) \quad \mathcal{A}(T) \subset \mathcal{A}(\mathcal{P}_E)$$

*Proof.* By Theorem 8 of Sec. 5.4 in [2], we have  $\mathcal{A}(J_f^E) \subset \mathcal{A}(\mathcal{P}_E)$  for any  $E$ -measurable complex function  $f$  on  $S$ . This implies that  $J_f^E \in \mathcal{A}(\mathcal{P}_E)$  for  $E$ -essentially bounded  $f$  because  $J_f^E$  belongs to  $L(\mathfrak{H})$  for such  $f$  and, hence, is contained in  $\mathcal{A}(J_f^E)$ . Conversely, Theorem 5 of Sec. 7.4 in [2] shows that any element of  $\mathcal{A}(\mathcal{P}_E)$  is equal to  $J_f^E$  for some  $E$ -measurable  $E$ -essentially bounded complex function  $f$  on  $S$ . It remains to prove that any closed densely defined operator  $T$  such that (8) holds is equal to  $J_f^E$  for some  $E$ -measurable complex function  $f$  on  $S$ . Since  $\mathcal{A}(\mathcal{P}_E)$  is Abelian, it follows from (8) that  $\mathcal{A}(T)$  is Abelian and, therefore,  $T$  is normal. Let  $\chi$  be a complex function on  $\mathbb{C}$  defined by the relation

$$\chi(z) = \frac{z}{|z| + 1}.$$

It is easy to see that the function  $\chi$  is one-to-one and maps  $\mathbb{C}$  onto the open unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Its inverse function  $\chi^{-1}$  is given by

$$\chi^{-1}(z) = \frac{z}{1-|z|}, \quad |z| < 1.$$

Since  $\chi$  is bounded and measurable on  $\mathbb{C}$ , we have  $\chi(T) \in \mathcal{A}(\mathcal{P}_{\mathcal{E}_T})$  and, hence,  $\chi(T) \in \mathcal{A}(T)$  because  $\mathcal{A}(\mathcal{P}_{\mathcal{E}_T}) = \mathcal{A}(T)$  (see Example 6). In view of (8), this implies that  $\chi(T) \in \mathcal{A}(\mathcal{P}_E)$  and, therefore,  $\chi(T) = J_g^E$  for some  $E$ -measurable function  $g$  on  $S$ . As  $\mathbb{C} \setminus \mathbb{D}$  is a  $\chi_*\mathcal{E}_T$ -null set and  $\chi^{-1}$  is a measurable map from  $\mathbb{D}$  to  $\mathbb{C}$ , the function  $\chi^{-1}$  is  $\chi_*\mathcal{E}_T$ -measurable on  $\mathbb{C}$ . By (6), we have  $\mathcal{E}_{\chi(T)} = \chi_*\mathcal{E}_T$  and, hence,  $\chi^{-1}$  is  $\mathcal{E}_{\chi(T)}$ -measurable. It therefore follows from (7) that

$$T = \chi^{-1}(\chi(T)) = \chi^{-1}(J_g^E) = J_f^E$$

for  $f = \chi^{-1} \circ g$ . The lemma is proved.  $\square$

Given a topological space  $S$ , we denote by  $C(S)$  the space of all continuous complex functions on  $S$ .

**Lemma 10.** *Let  $S$  be a Polish space,  $\mathfrak{H}$  be a Hilbert space, and  $E$  be a spectral measure for  $(S, \mathfrak{H})$ . Let  $\mathcal{C}$  be a subset of  $C(S)$  that separates the points of  $S$  and  $\mathcal{X}$  be the set of all operators  $J_f^E$  with  $f \in \mathcal{C}$ . Then  $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{P}_E)$ .*

In the proof below, all spectral integrals are taken with respect to  $E$ , and we write for brevity  $J_f$  instead of  $J_f^E$ .

*Proof.* Since  $\mathcal{A}(\mathcal{X}) \subset \mathcal{A}(\mathcal{P}_E)$  by Lemma 9, we have to show that  $\mathcal{A}(\mathcal{P}_E) \subset \mathcal{A}(\mathcal{X})$ .

Let  $\bar{\mathcal{C}}$  denote the set of functions, complex conjugate to the elements of  $\mathcal{C}$ , and let  $\mathfrak{A}$  be the subalgebra of  $C(S)$  generated by  $\mathcal{C} \cup \bar{\mathcal{C}}$  and the constant functions. Fix  $R \in (\mathcal{X} \cup \mathcal{X}^*)'$  and let  $\mathfrak{A}_R$  denote the subset of  $C(S)$  consisting of all  $f$  such that  $J_f$  commutes with  $R$ . If  $f, g \in \mathfrak{A}_R$ , then both  $J_f J_g$  and  $J_f + J_g$  commute with  $R$ . It follows from (3) that both  $J_{fg}$  and  $J_{f+g}$  commute<sup>3</sup> with  $R$ , i.e.,  $fg \in \mathfrak{A}_R$  and  $f + g \in \mathfrak{A}_R$ . Hence,  $\mathfrak{A}_R$  is an algebra. Since  $\mathfrak{A}_R$  obviously contains  $\mathcal{C} \cup \bar{\mathcal{C}}$  and all constant functions, we have  $\mathfrak{A}_R \supset \mathfrak{A}$ . Thus, every element of  $(\mathcal{X} \cup \mathcal{X}^*)'$  commutes with any operator  $J_f$  with  $f \in \mathfrak{A}$ .

Given  $f \in C(S)$  and a compact set  $K \subset S$ , we set  $B_{f,K} = J_f E(K)$ . Let  $\Psi \in \mathfrak{H}$  and  $\Phi = E(K)\Psi$ . Then for any measurable set  $A$ , we have  $E_\Phi(A) = E_\Psi(A \cap K)$  and, therefore,  $E_\Phi$  is a finite measure supported by  $K$ . In view of (2), this implies that  $\Phi \in D_{J_f}$ , i.e., the range of  $E(K)$  is contained in the domain of  $J_f$ . Since  $E(K) = J_{\chi_K}$ , where  $\chi_K$  is the characteristic function of  $K$ , it follows from (3) that  $B_{f,K} = J_{f\chi_K}$ . Hence,  $B_{f,K} \in L(\mathfrak{H})$  and (4) implies that

$$(9) \quad \|B_{f,K}\| \leq \sup_{s \in K} |f(s)|.$$

If  $f, g \in C(S)$ , then (3) implies that  $B_{f+g,K} = B_{f,K} + B_{g,K}$ .

We now show that

$$(10) \quad (\mathcal{X} \cup \mathcal{X}^*)' \subset \mathcal{Y}',$$

where  $\mathcal{Y}$  is the set of all  $J_f$  with  $f \in C(S)$ . Let  $R \in (\mathcal{X} \cup \mathcal{X}^*)'$ . We first prove that  $\mathcal{Y}'$  contains all operators  $R_K = E(K)RE(K)$ , where  $K$  is a compact subset of  $S$ . Fix  $f \in C(S)$  and let  $\varepsilon > 0$ . Since  $\mathcal{C} \subset \mathfrak{A}$ , the algebra  $\mathfrak{A}$  separates points

<sup>3</sup>If a closable densely defined operator  $T$  commutes with  $R \in L(\mathfrak{H})$ , then its closure  $\bar{T}$  also commutes with  $R$  because  $R = (R^*)^*$  and  $\bar{T} = (T^*)^*$ .

of  $S$ , and the Stone–Weierstrass theorem implies that there is  $g \in \mathfrak{A}$  such that  $|f(s) - g(s)| < \varepsilon$  for any  $s \in K$ . Since  $R_K$  commutes with both  $J_g$  and  $E(K)$ , it follows that  $R_K$  commutes with  $B_{g,K}$ . In view of (9), we have

$$\begin{aligned} \|B_{f,K}R_K - R_KB_{f,K}\| &\leq \|B_{f,K}R_K - B_{g,K}R_K\| + \|R_KB_{g,K} - R_KB_{f,K}\| \leq \\ &\leq 2\|B_{f-g,K}\|\|R\| < 2\varepsilon\|R\|. \end{aligned}$$

Because  $\varepsilon$  is arbitrary, this means that  $R_K$  commutes with  $B_{f,K}$ . This implies that  $R_K$  commutes with  $J_f$  because  $B_{f,K}R_K = J_fR_K$  and  $R_KB_{f,K}$  is an extension of  $R_KJ_f$  by the commutativity of  $E(K)$  and  $J_f$ . This proves that  $R_K \in \mathcal{Y}'$ . By Lemma 1,  $\mathcal{Y}'$  is strongly closed. Hence, inclusion (10) will be proved if we demonstrate that every strong neighborhood of  $R$  contains  $R_K$  for some compact set  $K$ . To this end, it suffices to show that for every  $\Psi \in \mathfrak{H}$  and  $\varepsilon > 0$ , there is a compact set  $K_{\Psi,\varepsilon}$  such that

$$(11) \quad \|(R - R_K)\Psi\| \leq \varepsilon$$

for any compact set  $K \supset K_{\Psi,\varepsilon}$ . Since

$$R - R_K = E(K)RE(S \setminus K) + E(S \setminus K)R,$$

we have

$$\|(R - R_K)\Psi\| \leq \|R\| E_{\Psi}(S \setminus K)^{\frac{1}{2}} + E_{\Phi}(S \setminus K)^{\frac{1}{2}},$$

where  $\Phi = R\Psi$ . As  $S$  is a Polish space, Theorem 1.3 in [1] ensures that there is a compact set  $K_{\Psi,\varepsilon}$  such that both  $E_{\Psi}(S \setminus K_{\Psi,\varepsilon})$  and  $E_{\Phi}(S \setminus K_{\Psi,\varepsilon})$  do not exceed  $\varepsilon^2/4$  and, therefore, (11) holds for any  $K \supset K_{\Psi,\varepsilon}$ . Inclusion (10) is thus proved.

We next show that

$$(12) \quad \mathcal{Y}' \subset \mathcal{P}'_E.$$

Let  $R \in \mathcal{Y}'$ . For any closed set  $F \subset S$ , it is easy to construct a uniformly bounded sequence of functions  $f_n \in C(S)$  that converges pointwise to  $\chi_F$ . Then  $J_{f_n}$  strongly converge to  $J_{\chi_F} = E(F)$  (see Theorem 2 of Sec. 5.3 in [2]). Since  $J_{f_n}$  commute with  $R$  for all  $n$ , this implies that  $E(F)$  commutes with  $R$ . Let  $\Sigma^R$  denote the set of all measurable sets  $A \subset S$  such that  $E(A)$  commutes with  $R$ . We have proved that  $\Sigma^R$  contains all closed sets. If  $A \in \Sigma^R$ , then  $E(S \setminus A) = 1 - E(A)$  commutes with  $R$  and, hence,  $S \setminus A \in \Sigma^R$ . If  $A_1, A_2 \in \Sigma^R$ , then both  $E(A_1 \cap A_2) = E(A_1)E(A_2)$  and  $E(A_1 \cup A_2) = E(A_1) + E(A_2) - E(A_1)E(A_2)$  commute with  $R$  and, therefore,  $A_1 \cap A_2$  and  $A_1 \cup A_2$  belong to  $\Sigma^R$ . Let  $A_n$  be a sequence of elements of  $\Sigma^R$  and  $A = \bigcup_{n=1}^{\infty} A_n$ . For all  $n = 1, 2, \dots$ , we set  $B_n = \bigcup_{j=1}^n A_j$ . Then  $B_n \in \Sigma^R$  for all  $n$ , and the  $\sigma$ -additivity of  $E$  implies that  $E(B_n)$  converge strongly to  $E(A)$ . Hence,  $E(A)$  commutes with  $R$ , i.e.,  $A \in \Sigma^R$ . We thus see that  $\Sigma^R$  is a  $\sigma$ -algebra containing all closed sets. This implies that  $\Sigma^R$  coincides with the Borel  $\sigma$ -algebra, and (12) is proved.

Inclusions (10) and (12) imply that  $(\mathcal{X} \cup \mathcal{X}')' \subset \mathcal{P}'_E$  and, hence,  $\mathcal{A}(\mathcal{P}_E) \subset \mathcal{A}(\mathcal{X})$ . The lemma is proved.  $\square$

The next lemma summarizes the facts about Polish and standard Borel spaces that are needed for the proof of Theorem 3.

**Lemma 11.** 1. *Let  $S$  and  $S'$  be standard Borel spaces and  $f: S \rightarrow S'$  be a one-to-one measurable mapping. Then  $f(S)$  is a measurable subset of  $S'$  and  $f$  is a measurable isomorphism from  $S$  onto  $f(S)$ .*

2. Let  $S$  be a Polish space and  $B$  be its Borel subset. Then there are a Polish space  $P$  and a continuous one-to-one map  $g: P \rightarrow S$  such that  $B = g(P)$ .
3. If  $S$  is a standard Borel space, then there exists a one-to-one measurable function from  $S$  to the segment  $[0, 1]$ .
4. Every measurable subset of a standard Borel space is itself a standard Borel space.

*Proof.* Statement 1 follows from Theorem 3.2 in [8], which, in its turn, is a reformulation of a theorem by Souslin (see [7], Chapter III, Sec. 35.IV). For the proof of statement 2, see Lemma 6 of Sec. IX.6.7 in [3]. To prove statement 3, we recall that every standard Borel space is either countable or isomorphic to the segment  $[0, 1]$  (see [9], Appendix, Corollary A.11). In the latter case, any isomorphism between  $S$  and  $[0, 1]$  gives us the required function. If  $S$  is countable, then we can just choose any one-to-one map from  $S$  to  $[0, 1]$  because all functions on  $S$  are measurable. Since every standard Borel space can be endowed with a Polish topology inducing its measurable structure, statement 4 follows from statements 1 and 2. The lemma is proved.  $\square$

**Lemma 12.** *Let  $E$  be a spectral measure on a standard Borel space  $S$ ,  $I$  be a countable set and  $\{f_\iota\}_{\iota \in I}$  be a family of measurable complex-valued functions on  $S$  that separates the points of  $S$ . Then the von Neumann algebra generated by all operators  $J_{f_\iota}$  with  $\iota \in I$  coincides with  $\mathcal{A}(\mathcal{P}_E)$ .*

*Proof.* Let  $f$  denote the map  $s \rightarrow \{f_\iota(s)\}_{\iota \in I}$  from  $S$  to  $\mathbb{C}^I$ . The space  $\mathbb{C}^I$  endowed with its natural product topology is a Polish space, and the measurability of  $f_\iota$  implies that of  $f$ . Since  $f_\iota$  separate the points of  $S$ , the map  $f$  is one-to-one. By statement 1 of Lemma 11,  $f(S)$  is a Borel subset of  $\mathbb{C}^I$  and  $f$  is a measurable isomorphism of  $S$  onto  $f(S)$ . By statement 2 of Lemma 11, there are a Polish space  $P$  and a continuous one-to-one map  $g: P \rightarrow \mathbb{C}^I$  such that  $f(S) = g(P)$ . Hence,  $h = f^{-1} \circ g$  is a measurable one-to-one map from  $P$  onto  $S$ . By statement 1 of Lemma 11,  $h$  is a measurable isomorphism from  $P$  onto  $S$ . We now use  $h$  to transfer the topology from  $P$  to  $S$ , i.e., we say that a set  $O \subset S$  is open if and only if  $h^{-1}(O)$  is open in  $P$ . Once  $S$  is equipped with this topology,  $h$  becomes a homeomorphism between  $P$  and  $S$  and, hence,  $S$  becomes a Polish space. Since  $h$  is a measurable isomorphism, the Borel measurable structure generated by the topology of  $S$  coincides with its initial measurable structure. Because  $f = g \circ h^{-1}$  is continuous, all  $f_\iota$  are continuous. Hence, the statement follows from Lemma 10. The lemma is proved.  $\square$

*Proof of Theorem 3.*

Suppose conditions (1) and (2) hold. By Lemma 9, condition (1) implies that  $\mathcal{A}(\mathcal{X}) \subset \mathcal{A}(\mathcal{P}_E)$ . Let the family  $\{f_\iota\}_{\iota \in I}$  be as in condition (2) and  $\mathcal{X}_0$  be the set of all  $J_{f_\iota}^E$  with  $\iota \in I$ . Without loss of generality, we can assume that all  $f_\iota$  are everywhere defined and measurable on  $S$ . Using statement 4 of Lemma 11 and the fact that  $E$  is standard, we can choose a measurable subset  $\tilde{S}$  of  $S$  such that  $E(S \setminus \tilde{S}) = 0$ , the family  $\{f_\iota\}_{\iota \in I}$  separates points of  $\tilde{S}$ , and  $\tilde{S}$ , considered as a measurable subspace of  $S$ , is a standard Borel space. Let  $\tilde{E}$  denote the restriction of  $E$  to  $\tilde{S}$ . For each  $\iota \in I$ , let  $\tilde{f}_\iota$  be the restriction of  $f_\iota$  to  $\tilde{S}$ . Then we have  $J_{\tilde{f}_\iota}^{\tilde{E}} = J_{f_\iota}^E$  for all  $\iota \in I$  and it follows from Lemma 12 that  $\mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{P}_{\tilde{E}})$ . As  $\mathcal{P}_E = \mathcal{P}_{\tilde{E}}$ , this implies that  $\mathcal{A}(\mathcal{P}_E) \subset \mathcal{A}(\mathcal{X})$  and, hence,  $\mathcal{A}(\mathcal{P}_E) = \mathcal{A}(\mathcal{X})$ .



Conversely, let  $\mathcal{A}(\mathcal{P}_E) = \mathcal{A}(\mathcal{X})$ . Then condition (1) is ensured by Lemma 9. By Lemma 8, there is a countable set  $\mathcal{X}_0 \subset \mathcal{X}$  such that  $\mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{X})$ . Choose a countable family  $\{f_\iota\}_{\iota \in I}$  of measurable complex functions on  $S$  such that each  $T \in \mathcal{X}_0$  is equal to  $J_{f_\iota}^E$  for some  $\iota \in I$ . It suffices to show that  $\{f_\iota\}_{\iota \in I}$  is  $E$ -separating. Let  $f$  be the measurable map  $s \rightarrow \{f_\iota(s)\}_{\iota \in I_0}$  from  $S$  to  $\mathbb{C}^I$ . For each  $\iota \in I$ , let  $\pi_\iota: \mathbb{C}^I \rightarrow \mathbb{C}$  be the function taking  $\{z_\kappa\}_{\kappa \in I}$  to  $z_\iota$ . For any  $\iota \in I$ , we have  $\pi_\iota \circ f = f_\iota$ , and it follows from (5) that  $J_{\pi_\iota}^{f_*E} = J_{f_\iota}^E$  for all  $\iota \in I$ , i.e., the set of all  $J_{\pi_\iota}^{f_*E}$  coincides with  $\mathcal{X}_0$ . Since the family  $\{\pi_\iota\}_{\iota \in I}$  obviously separates the points of  $\mathbb{C}^I$ , Lemma 10 implies that  $\mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{P}_{f_*E})$  and, hence,

$$(13) \quad \mathcal{A}(\mathcal{P}_E) = \mathcal{A}(\mathcal{P}_{f_*E}).$$

As above, let  $\tilde{S}$  be a measurable subset of  $S$  such that  $E(S \setminus \tilde{S}) = 0$  and  $\tilde{S}$ , considered as a measurable subspace of  $S$ , is a standard Borel space. By statement 3 of Lemma 11, there exists a one-to-one measurable function  $g$  on  $\tilde{S}$ . Clearly,  $g$  is  $E$ -measurable on  $S$  and it follows from Lemma 9 and (13) that  $\mathcal{A}(J_g^E) \subset \mathcal{A}(\mathcal{P}_{f_*E})$ . Now Lemma 9 implies that there exists a measurable function  $h$  on  $\mathbb{C}^I$  such that  $J_g^E = J_h^{f_*E}$ . In view of (5), this means that  $J_g^E = J_{h \circ f}^E$  and, hence,  $g$  and  $h \circ f$  are equal  $E$ -a.e.<sup>4</sup> Since  $g$  is one-to-one on  $\tilde{S}$ , it follows that  $f$  is one-to-one on  $\tilde{S} \setminus N$ , where  $N$  is the  $E$ -null set of all  $s \in \tilde{S}$  such that  $g(s) \neq h(f(s))$ . This means that  $\{f_\iota\}_{\iota \in I}$  is  $E$ -separating and the theorem is proved.  $\square$

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<sup>4</sup>We have  $J_f^E = 0$  for an  $E$ -measurable  $f$  if and only if  $f = 0$   $E$ -a.e. (see Lemma 2 of of Sec. 5.4 in [2]). By (3), this implies that  $J_f^E = J_g^E$  for  $E$ -measurable  $f$  and  $g$  if and only if  $f = g$   $E$ -a.e.